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On Mirror Symmetry for Manifolds of Exceptional Holonomy.

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Abstract

We consider Type II string theories on $\mathbf{T}^n/\mathbf{Z}_2^m$ Joyce orbifolds. This class contains orbifolds which can be desingularised to give manifolds of G_2 ($n=7$) and $Spin(7)$ holonomy ($n=8$). In the G_2 holonomy case we present two types of T -duality transformation which are clearly generalisations of the T -duality/mirror transformation in Calabi-Yau spaces. The first maps Type IIA theory on one such space from this class to Type IIB theory on another such space. The second maps Type IIA (IIB) to Type IIA (IIB). In the case of $Spin(7)$ holonomy we present a T -duality transformation which maps Type IIA (IIB) theory on one such space to Type IIA (IIB) on another such space. As orbifold conformal field theories these T -dual target spaces are related via the inclusion/exclusion of discrete torsion and the T -duality is proven to genus g in string perturbation theory. We then apply a Strominger, Yau, Zaslow type argument which suggests that manifolds of G_2 holonomy which have a “mirror” of the first (second) type admit supersymmetric \mathbf{T}^3 (\mathbf{T}^4) fibrations and that manifolds of $Spin(7)$ holonomy for which a mirror exists admit fibration by supersymmetric 4-tori. Further evidence for this suggestion is given by examining the moduli space structure of wrapped D-branes.

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1 Introduction.

Mirror symmetry has proven to be a powerful and fascinating area of string theory (for a review see [1]). Mirror symmetry is at present only defined for string theory on Calabi-Yau spaces. Recent work [2, 3, 4] has suggested the possible existence of an analogous symmetry for string theory on manifolds of exceptional holonomy. This evidence is perhaps not surprising since manifolds of exceptional holonomy share many properties with Calabi-Yau manifolds. They admit Ricci flat metrics and covariantly constant forms leading to the important fact that they give supersymmetric compactifications of string theory and M theory. Another remarkable general feature is that Euclidean D-branes wrapped around supersymmetric cycles in these manifolds have world-volume theories which are cohomological [5, 6, 7, 8, 9]. This latter feature is almost certainly related to what we will have to say in this paper, but will not be discussed here.

String theories on orbifold limits of such manifolds can provide a simple framework in which to study a variety of physical and mathematical phenomena which often persist as one moves away from the orbifold limit. This is due to the fact that strings remarkably “know how to” blow up the orbifold. We will exploit these simplifications here.

The purpose of this paper is to provide evidence for the existence of a symmetry in Type IIA/IIB string theory compactified on manifolds of exceptional holonomy. As we will see, this symmetry is very much analogous to the mirror symmetry of Type II strings on Calabi-Yau n -folds.

In the case of Calabi-Yau threefolds, Strominger, Yau and Zaslow (SYZ) considered some non-perturbative implications of mirror symmetry [10]. By considering the mirror BPS soliton spectra in the two theories they argued that mirror symmetry is T -duality. The analogue of their argument for Calabi-Yau n -folds was given in [13]. The outcome of the SYZ argument is that Calabi-Yau n -folds which have a mirror admit supersymmetric \mathbf{T}^n fibrations. Mirror symmetry is then T -duality on these supersymmetric fibers.

Owing to the simplifications in studying orbifolds, one well understood example of mirror symmetry is the pair of $\mathbf{T}^6/\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds first constructed in [11]. This pair of orbifolds differ by the inclusion/exclusion of

discrete torsion². In this case, for each choice of orbifold isometry group, there is only one consistent choice for the discrete torsion leading to the fact that there are only two such orbifolds, one without discrete torsion and the other with. As pointed out in [11] this pair of orbifolds have mirror Hodge numbers. Let us denote these two target spaces as \mathbf{X} and \mathbf{Y} . This pair of orbifolds were further studied in [14] where it was shown that Type IIA theory on \mathbf{X} is equivalent to Type IIB theory on \mathbf{Y} . This equivalence was proven to all orders in string perturbation theory. Moreover the mirror symmetry transformation between these two theories was shown to be T -duality. This example thus provides strong evidence that the SYZ argument is correct³.

In [16] we tested the SYZ argument for Calabi-Yau fourfolds by studying $\mathbf{T}^8/\mathbf{Z}_2$ ³ orbifolds. The mirror transformation is also $R \rightarrow 1/R$ T -duality, but this time on supersymmetric \mathbf{T}^4 fibers. In this case also, it is the turning on of discrete torsion via the T -duality transformation which “produces” the mirror target space. As in [14] equivalence between these mirror theories can be checked to genus g in string perturbation theory.

In [20, 21, 22] Joyce showed that manifolds with G_2 and $Spin(7)$ holonomy can also be constructed by desingularising $\mathbf{T}^n/\mathbf{Z}_2^m$ orbifolds, where $n = 7$ and 8 for G_2 and $Spin(7)$ holonomy respectively. We studied Type II theories on such orbifolds (for $n = 7$) in [4] where we proved that Type IIA theory on a given orbifold from this class is equivalent to Type IIB theory on another orbifold from the same class. This verified conjectures made in [2, 3]. Again, these examples follow the same pattern: the transformation between the two theories is T -duality on supersymmetric tori, they differ via the inclusion/exclusion of discrete torsion and the equivalence can be checked to all orders in string perturbation theory. In this sense, this transformation between Type IIA and IIB theories on spaces of G_2 holonomy can be regarded as a direct generalisation of mirror symmetry for Calabi-Yau n -folds. In this paper we will further develop the notion of a “mirror” symmetry for manifolds of exceptional holonomy. The plan of the paper is as follows.

Motivated by the results which follow we first give a simple definition of mirror manifolds in Type II string theory. This definition is a very general one and, for the cases which we study here, is likely to be replaced with a much more specific conjecture in the future. The results of this paper are

²Discrete torsion was defined in [12].

³Further aspects of this mirror pair were studied in [15] in light of [10].

independent of this definition, but we make this definition for two reasons: (*i*): it allows a simplified presentation of the results; (*ii*): since, in the examples studied here, a symmetry analogous to mirror symmetry for Calabi-Yau spaces exists, it deserves to have a name. For obvious reasons we choose to call it exceptional mirror symmetry, or just mirror symmetry when the context is obvious. Throughout this paper a generic pair of mirror target spaces will be denoted by **X** and **Y** respectively.

In section three we study Type II strings on $\mathbf{T}^7/\mathbf{Z}_2^n$ orbifolds. This class of orbifolds includes those which can be blown up to give manifolds of G_2 holonomy [20, 21]. We describe two types of T -duality/mirror transformations which map between pairs of target spaces in this class. The first are of the type studied in [4] and map Type IIA on one target space **X** from this category to Type IIB on the mirror **Y**, where the mirror is defined in the previous section. The second type of transformation maps Type IIA on **X** to Type IIA on **Y** and similarly maps IIB to IIB.

In section four we study Type II theories on $\mathbf{T}^8/\mathbf{Z}_2^n$ orbifolds, some of which admit a resolution to manifolds of $Spin(7)$ holonomy. We present a T -duality/mirror transformation which maps Type IIA (IIB) theory on one such space (**X**) to Type IIA (IIB) theory on the mirror **Y**.

In section five we consider the Strominger, Yau, Zaslow argument [10] applied to all these three cases. Even though the results are not as strong as in the Calabi-Yau cases, the results suggest that manifolds of G_2 holonomy with a mirror of the first type admit supersymmetric \mathbf{T}^3 fibrations and that those with a mirror of the second type admit supersymmetric \mathbf{T}^4 fibrations. Similarly, it is conjectured that manifolds of $Spin(7)$ holonomy for which a mirror exists admit supersymmetric \mathbf{T}^4 fibrations.

In section six we give further evidence for the conjectures of section five by considering the geometry of the classical orbifolds themselves. We first point out that all the orbifolds we have discussed indeed have the predicted fibration structure. We then show that the moduli space of the relevant D-brane wrapped around the corresponding p -torus is also an orbifold in the class we are considering.

2 A Definition of Mirror Symmetry.

As we have discussed in the introduction, we will describe in this paper a symmetry analogous to mirror symmetry for Calabi-Yau n -folds in Type II string theories on manifolds of exceptional holonomy. In order to make the presentation of these results clearer as well as to give this symmetry a name it is useful to give a definition of such a symmetry here. Mirror symmetry in the conventional sense applies to two Type II theories compactified on different manifolds of $SU(n)$ holonomy. Both compactifications yield isomorphic physics. The perturbation theories on the mirror manifolds are also directly related. Motivated by these observations, we will study the following circumstances: Type IIR string theory (where R can be A or B) compactified on a manifold \mathbf{X}_d yields completely isomorphic physics to Type IIS string theory (S can be A or B) compactified on the manifold \mathbf{Y}_d . The two perturbation theories are directly related under the map from one theory to the other. Here, R and S can be either A or B depending upon the circumstances, and in general \mathbf{X} is a topologically different manifold from \mathbf{Y} . Both manifolds are d -dimensional.

Definition: When two theories satisfying the above criteria exist we will say that the two theories are mirror, and that \mathbf{X} and \mathbf{Y} are mirror manifolds.

We will now restrict our attention to the case when both mirror compactifications preserve some fraction of the supersymmetries of the ten-dimensional theory. This means that both \mathbf{X}_d and \mathbf{Y}_d are Ricci flat manifolds of “special” holonomy⁴. Both \mathbf{X} and \mathbf{Y} must have the same holonomy as we do not expect two theories with different numbers of supersymmetries to yield isomorphic physics. We will be interested in the cases when both manifolds have holonomy G_2 or $Spin(7)$. In these cases, if the need arises to differentiate between the mirror symmetry discussed here and that for Calabi-Yau spaces we can call the symmetry discussed here “exceptional mirror symmetry”.

Note that the term “mirror symmetry” originates from the relationship between Hodge diamonds in mirror Calabi-Yau spaces. Even though we choose the name mirror symmetry for the analogous symmetry discussed here for manifolds of exceptional holonomy, the analogue of the Hodge diamonds in the “mirror” target spaces are not related in any similar fashion [4].

⁴By “special” we mean $SU(n)$, G_2 or $Spin(7)$.

3 T -duality as Mirror Symmetry for Joyce 7-Orbifolds.

In this section we will study Type IIA (or IIB) string theory defined on a class of Joyce orbifolds. Some of the orbifolds in this class (perhaps all) can be desingularised to give smooth manifolds of G_2 holonomy. However, regardless of whether some “classical” interpretation for these target spaces exists, these orbifolds are well defined in string theory. This class of orbifolds will be defined as orbifolds of the form \mathbf{T}^7/Γ , where $\Gamma \cong \mathbf{Z}_2^n$. Three of these \mathbf{Z}_2 generators act non-freely on the torus; the remaining act freely and do not break any supersymmetry.

The simplest orbifold in this category may be defined as follows.

Consider the seven-torus, \mathbf{T}^7 , with coordinates x_1, x_2, \dots, x_7 . Define the \mathbf{Z}_2^3 orbifold group Γ , with generators α, β, γ as follows:

$$\alpha(x_i) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \quad (1)$$

$$\beta(x_i) = (-x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7) \quad (2)$$

$$\gamma(x_i) = (-x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7) \quad (3)$$

In [4], we showed that under the T -duality transformation which inverts the radii of the \mathbf{T}^3 with coordinates x_2, x_3, x_5 , Type IIA (IIB) theory on this orbifold is mapped to Type IIB (IIA) theory on the same orbifold⁵, but with a certain “amount” of discrete torsion turned on. Equivalence between these dual theories was checked to genus g in the string path integral. Let us denote the mirror transformation as T_{235} . This is the first of two types of mirror transformations we will discuss in G_2 holonomy compactifications of Type II theories.

The assertion that the two theories are equivalent follows from the following formula for how the genus g path integral measure transforms under T_{235} :

$$\mu_{g,\alpha} \longrightarrow (-1)^{\sigma_\alpha} \cdot \epsilon \cdot \mu_{g,\alpha} \quad (4)$$

Here, $\mu_{g,\alpha}$ is the path integral measure for the original theory, at genus g . The genus g Riemann surface has spin structure α . σ_α is the parity of

⁵by the same orbifold we mean the orbifold with the same generators, but of course with T -dual radii.

this Riemann surface. The factor corresponding to the parity which appears in the “mirror” measure is responsible for converting IIA to IIB and vice-versa. Finally, ϵ is the discrete torsion in the “mirror” theory. This formula was derived in [4] using the techniques developed in [14] and the explanation of why the proof of this formula is a proof of equivalence between the two theories is also explained in [14]. Following the notations of the last section, let us denote the target space of the undualised theory as \mathbf{X} and that of the mirror theory as \mathbf{Y} . Since the above formula applies to all Joyce orbifolds of the form $\mathbf{T}^7/\mathbf{Z}_2^n$, we are free to restrict our attention to the subclass which contains those which admit desingularisation to manifolds of G_2 holonomy. We may do this in order to discuss the implications of this “mirror” symmetry for the structure of bona fide manifolds of G_2 holonomy. However our considerations also apply to any of the more stringy target spaces which might exist in the class of orbifolds we are considering.

If we take the original theory to be Type IIA on \mathbf{X} , then the moduli space of the zero-brane⁶ is \mathbf{X} (here and throughout this paper we make the assumption that all T -duality symmetries are exact.). Under T_{235} , this brane is mapped to the 3-brane of Type IIB theory wrapped around a supersymmetric \mathbf{T}^3 in \mathbf{Y} . Because the target spaces \mathbf{X} and \mathbf{Y} differ only by the inclusion of discrete torsion, their untwisted sectors are the same. Geometrically this sector is associated with the elements of homology/cohomology of \mathbf{T}^7 , which are invariant under the orbifold group. Because the mirror transformation can be applied at the orbifold limit of the theory, the superymmetric \mathbf{T}^3 around which the 3-brane is wrapped should correspond to a \mathbf{T}^3 submanifold of \mathbf{T}^7 invariant under the orbifold isometry group. Moreover, with the choice of coordinates given above, this \mathbf{T}^3 has local coordinates x_2, x_3, x_5 . Let us denote this torus by \mathbf{T}^3_{235} . In fact, at a given point in moduli space, \mathbf{T}^3_{235} is one of only seven supersymmetric 3-tori present in the untwisted sector of the orbifold. An easy way to see this is that there exists on $\mathbf{T}^7/\mathbf{Z}_2^n$ a covariantly constant 3-form, ϕ . Its components are given by⁷:

$$\phi = dx_{127} + dx_{136} + dx_{145} + dx_{235} - dx_{246} + dx_{347} + dx_{567} \quad (5)$$

ϕ , being G_2 invariant encodes the holonomy structure of the target space. The restriction of ϕ to any of the invariant 3-tori in \mathbf{T}^7 is the volume form on

⁶All the branes we discuss in this paper are D-branes. We drop the “D” for simplicity.

⁷ dx_{ijk} means $dx_i \wedge dx_j \wedge dx_k$.

that 3-cycle. Such cycles are minimal volume [18] and hence supersymmetric (BPS).

One can consider T -dualising these other tori, and one indeed finds other examples of mirror pairs of theories. It would be interesting to compute the Betti numbers of these other mirror pairs of target spaces.

Since in these examples one has a concrete understanding that the two theories are mirror, it must be true that the moduli space of the 3-brane wrapped around \mathbf{T}^3_{235} in \mathbf{Y} is \mathbf{X} . This result gives a prediction for the dimension of the space of supersymmetric deformations of 3-tori in manifolds of G_2 holonomy, which we will discuss in sections five and six.

We now give an example which involves T -dualising a supersymmetric four-torus in the same class of Joyce orbifolds. This is the second type of mirror symmetry transformation which we mentioned earlier. Consider the supersymmetric four-torus, \mathbf{T}^4_{1467} , where the subscript indicates the coordinates of this cycle.

Following the techniques in [14], it can be shown that under the T -duality transformation (T_{1467}) which inverts the radii of this torus, the path integral measure of the Type IIA or IIB theory transforms as follows:

$$\mu_{g,\alpha} \longrightarrow \epsilon \cdot \mu_{g,\alpha} \quad (6)$$

This formula shows that for all the target spaces in the category we are considering, Type IIA (or IIB) theory on the orbifold without discrete torsion transforms into Type IIA (or IIB) theory on the same orbifold, but with discrete torsion turned on. Furthermore, if we began with Type IIA theory on \mathbf{X} , the zero-brane is mapped to a 4-brane in IIA theory on \mathbf{Y} wrapped around \mathbf{T}^4_{1467} . The moduli space of this brane is \mathbf{X} . In sections five and six we discuss this moduli space. Since the above results apply to a large number of mirror target spaces, this is strong evidence that the SYZ argument can be successfully applied to mirror manifolds of G_2 holonomy. As was the case for the supersymmetric 3-tori, in this class of orbifolds there also exist seven supersymmetric 4-tori in the “untwisted sector” of the orbifold, of which \mathbf{T}^4_{1467} is just one example. In fact these seven 4-tori are just the Hodge-Poincare duals of the seven supersymmetric 3-tori. T -dualising these other six 4-tori gives further examples of mirror target spaces.

It is clear from our introductory discussion that these mirror transformations are direct analogues of the mirror symmetry transformations in $\mathbf{T}^m/\mathbf{Z}_2^n$

Calabi-Yau orbifolds.

4 Joyce 8-orbifolds.

We can also give examples of mirror target spaces of $Spin(7)$ holonomy. These are based on the Joyce orbifolds considered in [22]. The class of such orbifolds that we will consider are all orbifolds of the form \mathbf{T}^8/Γ . Here $\Gamma \cong \mathbf{Z}_2^n$. Four of the \mathbf{Z}_2 generators of Γ each break half the supersymmetries present in compactification on \mathbf{T}^8 , ie together they preserve 1/16 of the original supersymmetry. The remaining generators act freely on the torus and break no supersymmetry. The simplest example in this category can be defined as follows. Compactify the example above (eqs. (1) – (3)) on a further \mathbf{S}^1 with coordinate x_8 , and orbifold the theory by a further \mathbf{Z}_2 isometry which acts in the following way:

$$\delta(x_i) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8) \quad (7)$$

We are thus considering the orbifold of \mathbf{T}^8 with coordinates (x_1, \dots, x_8) and generators $(\alpha, \beta, \gamma, \delta)$. In the untwisted sector of any of the orbifolds in this class, there exist 14 supersymmetric 4-tori. However, Hodge duality interchanges the volume forms of these tori, giving seven pairs of dual 4-tori. The volume forms are given by the components of the $Spin(7)$ invariant 4-form. Our choice of coordinates coincides with that of Joyce [22], and the components of this 4-form are given in [22]. Let us denote these 14 supersymmetric 4-tori by \mathbf{T}^4_i .

It may then be checked that upon T -dualising \mathbf{T}^4_i , the path integral measure of the Type IIA or IIB theory transforms in the following way:

$$\mu_{g,\alpha} \longrightarrow \epsilon_i \cdot \mu_{g,\alpha} \quad (8)$$

Here, ϵ_i is the discrete torsion associated with the i 'th mirror transformation. The discrete torsion in each of the fourteen cases can be calculated following the methods of [14]. This formula shows that under the fourteen mirror symmetry transformations Type IIA (or IIB) theory on \mathbf{X} transforms into Type IIA (or IIB) theory on \mathbf{Y} ; where \mathbf{X} and \mathbf{Y} are both orbifold target spaces from the class of 8-orbifolds we are restricting our attention to. As orbifold conformal field theories, \mathbf{X} and \mathbf{Y} differ by the discrete torsion factor ϵ_i . Clearly, by considering the zero-brane whose moduli space is \mathbf{X} (\mathbf{Y}),

under all these mirror transformations, this brane is mapped to a 4-brane wrapped around \mathbf{T}^4_i . This result thus makes a prediction about the moduli space of supersymmetric deformations of a supersymmetric 4-torus in a manifold of $Spin(7)$ holonomy, which we will consider shortly.

5 Toroidal Fibrations and Mirror Symmetry.

In this section we study the generalisation of the Strominger, Yau, Zaslow argument to mirror manifolds of exceptional holonomy. Since the Type II theories have $N=2$ supersymmetry in ten dimensions, when compactified on a manifold of special holonomy, the resulting $(10-d)$ -dimensional theory also has $N=2$ supersymmetry. In particular, this means that Dirichlet p -branes which are reduced from ten dimensions break half of these supersymmetries and are BPS states. Because the perturbation theories of the two mirror theories are directly related (as is the case for mirror Calabi-Yau threefold compactifications), we can compare the BPS soliton spectra in both theories. In particular we can expect the moduli spaces of BPS branes exchanged by the mirror transformation to be identical. (In order for this to be true the mirror symmetry must be an exact non-perturbative symmetry. The T -dualities of the previous sections are exact perturbatively. However we expect T -dualities to be exact dualities of the full theory and not just an artefact of string perturbation theory. This structure is of course required by U -duality [23]). In general, as pointed out in [10] there will be world-sheet instanton corrections to the moduli spaces of wrapped branes. However in this paper we will neglect these corrections as we wish to focus on the general structure predicted by exceptional mirror symmetry. As in [10] these corrections will be important in understanding the precise details of the picture which emerges.

Let us now consider applying the SYZ argument with the assumption that a pair of mirror Type II theories exists. We can begin by considering Type IIA theory on $\mathbf{R}^{10-d} \times \mathbf{X}_d$. The starting point for applying the SYZ argument is the observation that the moduli space of the zero-brane, dimensionally reduced from ten dimensions, is \mathbf{X}_d . We then look for this zero-brane in the mirror theory, which is either Type IIA or IIB theory on \mathbf{Y}_d . In the case where the mirror theory is Type IIA on \mathbf{Y}_d , we can not identify the dimensionally reduced zero-brane as the brane we are looking for because its moduli space is \mathbf{Y}_d and \mathbf{Y}_d and \mathbf{X}_d are different manifolds. We may therefore

conclude that the zero-brane in the string theory on \mathbf{Y}_d is a Dirichlet p -brane which is wrapped around a supersymmetric p -cycle [17], $C_p \subset \mathbf{Y}_d$. Because the two theories are mirror, the moduli space of this wrapped p -brane is \mathbf{X}_d .

Let us consider the structure of the moduli space of the wrapped p -brane world-volume theory. We are lead to consider the “compactification” of this $p+1$ dimensional theory on C_p , to $0+1$ dimensions. The theory contains a $U(1)$ gauge field. Reducing this gauge field on 1-cycles of C_p gives $b_1(C_p)$ scalars in the reduced world volume theory. With all other moduli fixed, the moduli space of these scalars is the torus \mathbf{T}^{b_1} . The other moduli in the theory correspond to supersymmetric, normal deformations of C_p within \mathbf{Y}_d . We will call this moduli space the space of supersymmetric deformations and denote it by \mathbf{B}_c , where c is its dimension. At a generic⁸ point on \mathbf{B}_c , we have a torus \mathbf{T}^{b_1} . We can therefore see that the moduli space of the wrapped p -brane theory has the structure of a fibration with base \mathbf{B}_c and fiber \mathbf{T}^{b_1} . This is by no means a precise statement. However, intuitively this is the natural picture which emerges. This structure has been useful in similar, related discussions of wrapped branes [10, 5]. Furthermore, the later arguments that we will give provide further evidence that this is indeed the structure of the moduli space at generic points. For now we will assume that this fibration picture of the moduli space is correct.

Since the space in question is \mathbf{X}_d , we require

$$c + b_1(C_p) = d \quad (9)$$

We can now restrict our attentions to the cases when both \mathbf{X} and \mathbf{Y} have exceptional holonomy.

Since C_p is a supersymmetric cycle, it is a calibrated submanifold of \mathbf{Y}_d . This means that there exists on \mathbf{Y}_d a closed exterior p -form (ϕ_p) whose restriction to C_p is the volume form of C_p [18]. On 7-manifolds of G_2 holonomy there exists a covariantly constant 3-form whose Hodge dual is a covariantly constant 4-form. The two natural classes of supersymmetric cycles in G_2 holonomy manifolds are therefore 3-cycles and 4-cycles. In fact, these are the only known classes of supersymmetric cycles on such manifolds. For compactification on manifolds of $Spin(7)$ holonomy, the only known class of supersymmetric cycles are 4-cycles, which are calibrated by the covariantly

⁸We say generic because there will in general exist points on the base at which the fiber degenerates.

constant 4-form, which encodes the holonomy structure of the manifold. The fact that the calibrated submanifolds of manifolds of exceptional holonomy are supersymmetric cycles was shown in [5, 13].

Let us consider the 4-cycles in the case when both \mathbf{X} and \mathbf{Y} are 7-manifolds of G_2 holonomy. Mclean [19] has shown in this case that

$$c = b_2^+(C_4) \quad (10)$$

where b_2^+ denotes the number of self-dual harmonic 2-forms.

We can now consider equation (9). This becomes

$$b_2^+(C_4) + b_1(C_4) = 7 \quad (11)$$

The only 4-manifold we know, which satisfies this constraint, is the 4-torus \mathbf{T}^4 . It is not known to us whether or not this is the only 4-manifold for which this is true. However, assuming that this is the case, our zero-brane in Type IIA theory on \mathbf{X}_7 may be identified with a 4-brane wrapped around a supersymmetric \mathbf{T}^4 in \mathbf{Y}_7 . Furthermore, the moduli space of this 4-brane (which we identify with \mathbf{X}_7) is a $\mathbf{T}^{4'}$ fibration over \mathbf{B}_3 (where $\mathbf{T}^{4'}$ is the 4-torus which is the moduli space of the gauge connections). Moreover the zero-brane in the Type IIA theory on \mathbf{Y}_7 can be identified with a 4-brane in the theory on \mathbf{X}_7 , wrapped around another supersymmetric \mathbf{T}^4 . We therefore learn that both \mathbf{X}_7 and \mathbf{Y}_7 contain supersymmetric 4-tori and are (at least locally) fibred by these 4-tori.

We can then identify the mirror symmetry transformation as the $R \rightarrow 1/R$ T -duality transformation on the \mathbf{T}^4 fibres, since we know that the zero-brane and 4-brane are interchanged under such a transformation. This implies that the \mathbf{T}^4 fibers of \mathbf{X} are T -dual to those of \mathbf{Y} . The results of section three provide strong evidence that the SYZ argument has been successfully applied here.

To summarise the results so far, we have learned that in Type IIA compactification on a 7-manifold of G_2 holonomy (\mathbf{X}), the existence of a mirror Type IIA theory on the mirror manifold of G_2 holonomy (\mathbf{Y}) leads to the conclusion that both \mathbf{X} and \mathbf{Y} admit supersymmetric \mathbf{T}^4 fibrations. The mirror transformation is T -duality on the \mathbf{T}^4 fibers. Note that strictly speaking this conclusion may only be drawn if \mathbf{T}^4 is the only 4-manifold which satisfies equation (11).

So what about the 3-cycles in the G_2 case and the 4-cycles in the $Spin(7)$ case? Unfortunately, in these two cases, c is difficult to determine, because it is not known to be a topological invariant of the supersymmetric cycle. In fact, in these cases, c is the number of harmonic spinors with values in a particular vector bundle, whose definition may be found in [19]. In order to deal with this situation we will make further assumptions.

Let us first discuss the general picture which is emerging. For the case of Calabi-Yau n -folds and the case of 7-manifolds of G_2 holonomy which we have just discussed, the SYZ argument suggests that when such manifolds have mirrors, they are fibered by supersymmetric tori. The ultimate reason that this is so is that mirror symmetry is T -duality. Thus, the picture emerging is that of mirror manifolds with supersymmetric cycles which fiber the whole manifold. It is natural to hope that a similar story will emerge for the 3-cycles on 7-manifolds of G_2 holonomy and for the 4-cycles on manifolds of $Spin(7)$ holonomy.

With this in mind, consider the case in which the supersymmetric cycle C_p , locally fibers the whole manifold \mathbf{Y}_d . In this case, the space of supersymmetric deformations has dimension

$$c = d - p \quad (12)$$

Let us assume that this is the case and consider equation (9) again. This now says that

$$b_1(C_p) = p \quad (13)$$

This equation has a natural solution: C_p is a flat p -torus \mathbf{T}^p . What this shows is that if there exists a supersymmetric cycle C_p in \mathbf{Y}_d which locally fibers all of \mathbf{Y}_d , then the moduli space of the p -brane wrapped around this cycle has the same dimension as \mathbf{X}_d if $C_p \cong \mathbf{T}^p$. Thus if the original zero-brane is to be identified with a p -brane wrapped around a cycle which locally fibers all of \mathbf{Y}_d , then this identification is possible if this cycle is a p -torus.

This simple result is quite remarkable since this is the general feature which appears to emerge from the SYZ argument which ultimately results in the statement that mirror symmetry is T -duality on such tori.

It follows that if the dimension of \mathbf{B}_e for a supersymmetric \mathbf{T}^3 (resp. \mathbf{T}^4) submanifold of a manifold of G_2 holonomy (resp. $Spin(7)$ holonomy) is indeed four (resp. four), then by the SYZ argument we should conclude

that manifolds of G_2 ($Spin(7)$) holonomy which have mirrors admit \mathbf{T}^3 (\mathbf{T}^4) fibrations.

The SYZ argument given in this section has been applied completely independently to the results of the previous sections; we simply assumed the existence of mirror compactifications of the Type II theories, according to the definition given earlier. If we now consider our earlier results then we see that they provide strong evidence for the conclusions drawn here. In particular, in the case of supersymmetric 3-tori (4-tori) in manifolds of G_2 ($Spin(7)$) holonomy, the results of sections three and four require the space of supersymmetric deformations, \mathbf{B}_c , to be four dimensional in each case. Combining these results with those in [10, 13] leads to the general picture that manifolds with $SU(n)$, G_2 or $Spin(7)$ holonomy for which a mirror exists admit supersymmetric \mathbf{T}^p fibrations with p the dimension of the fiber varying from case to case. Moreover, mirror symmetry is T -duality on these toroidal fibers.

6 Moduli Space Structure.

The results of the previous sections have suggested a particular fibration structure for manifolds of exceptional holonomy. Furthermore the moduli spaces of certain wrapped D-branes should also share these properties. In this section we will again appeal to the simplifying features of orbifolds. The target spaces we have considered here contain manifolds of exceptional holonomy which possess an orbifold limit. In particular we can expect the orbifold, viewed as a space in classical geometry to inherit this fibration structure. This is because the arguments we have given apply in principle at any point in the full moduli space of the string compactification, and the “blowing up” modes are moduli in this space. By considering the classical orbifolds themselves we will indeed see that the correct structure emerges.

We begin with one of the Type II theories on a d -torus, $\mathbf{T}^d \cong \mathbf{T}^p \times \mathbf{T}^{d-p}$. Now we take an orbifold of this theory. Denote the finite orbifold isometry group as Γ . In general, each element of Γ will have a piece which acts on \mathbf{T}^p and a piece which acts on \mathbf{T}^{d-p} . We may write $\Gamma \cong h_p.h_{d-p}$, where the subscripts denote the pieces of Γ which act on the tori of the corresponding dimension.

We will denote the orbifold we are considering as

$$\mathbf{M}_\Gamma = (\mathbf{T}^p \times \mathbf{T}^{d-p}) / (h_p.h_{d-p}). \quad (14)$$

We will assume that the volume form of \mathbf{T}^p is preserved by Γ so that \mathbf{T}^p is a supersymmetric cycle in the orbifold. Since the Hodge dual of this form is the volume form of \mathbf{T}^{d-p} , \mathbf{T}^{d-p} is also preserved by the orbifold group. Away from the fixed points of h_{d-p} in \mathbf{T}^{d-p} , we can view the whole space as a fiber bundle with fiber \mathbf{T}^p and base \mathbf{T}^{d-p}/h_{d-p} [24]. This description breaks down at the the fixed points in \mathbf{T}^{d-p} where the fibre “degenerates” to \mathbf{T}^p/h_p .

The description we have just given applies to the Joyce 7- and 8-orbifolds of the preceding section. Moreover, in the case of the 7-orbifolds the *only* Γ invariant cycles are 3-tori and 4-tori (apart from the zero cycle and the fundamental cycle). Similarly, the only Γ invariant 4-cycles in the 8-orbifolds are 4-tori. Thus, away from the “degeneration points”, the 7-orbifolds have the structure of fibrations with supersymmetric \mathbf{T}^3 (or \mathbf{T}^4) fibres and \mathbf{T}^4/h_4 (\mathbf{T}^3/h_3) base. Similarly, the 8-orbifolds have the structure of fibrations with supersymmetric \mathbf{T}^4 fibres. Thus the classical orbifolds themselves have the

structure predicted by our application of the SYZ argument. We expect that desingularisation of the orbifold leaves this structure intact although it is difficult to make a precise statement about this. We will now apply an argument which shows that the moduli space of the world volume theory of a p -brane wrapped around these fibers has precisely the same structure.

Consider the adiabatic limit in which \mathbf{T}^{d-p}/h_{d-p} is very large, or equivalently that \mathbf{T}^p varies slowly over the base. In this adiabatic limit we can consider a p -brane wrapped around the supersymmetric fiber \mathbf{T}^p . In this limit, the brane is generically away from the “degeneration” points. Thus, as far as the adiabatic brane is concerned, it is wrapped around \mathbf{T}^p and moves on \mathbf{R}^{d-p} . In this limit, under T -duality on \mathbf{T}^p , the p -brane is mapped to a zero brane in Type IIA theory compactified on the T -dual torus to \mathbf{T}^p . As time goes on the wrapped brane eventually discovers the space of all of the normal, supersymmetric deformations. Away from the degeneration points where the cycle degenerates, the moduli space of the p -brane theory is thus a fiber bundle with base \mathbf{T}^{d-p}/h_{d-p} . The fibre in the moduli space is a p -torus \mathbf{T}^p' . This moduli space is d -dimensional if we assume that the dimension does not change at the degeneration points. This is a natural assumption if we require the physics to remain non-singular.

Clearly, by applying this adiabatic argument to a brane wrapped around a supersymmetric cycle in any of the Joyce orbifolds of the preceding section, we see that its moduli space has the same structure as the orbifolds themselves. This gives us further confidence in our application of the SYZ argument since we indeed wanted to identify the moduli space of the brane itself as a Joyce orbifold. Now let us consider the cases in which the target space orbifold admits a desingularisation to a manifold of exceptional holonomy. Clearly, desingularising the target space orbifold induces a desingularisation of the wrapped brane moduli space orbifold via the T -duality/mirror transformation. We thus find that both the target space and the wrapped brane moduli space are manifolds of exceptional holonomy. In fact by considering Type IIA theory on \mathbf{M}_Γ and applying the relevant T -duality/mirror transformation, the moduli space of the relevant wrapped D-brane is topologically the same classical orbifold as \mathbf{M}_Γ . This statement may be verified by noting that the action of Γ on the torus in the p -brane moduli space is precisely the same as its action on the torus in \mathbf{M}_Γ .

7 Summary and Outlook.

We have applied the SYZ argument to mirror Type II theories on manifolds of exceptional holonomy. This has lead us to the following conclusions: If Type IIA theory on a manifold of G_2 holonomy is mirror to Type IIA (IIB) theory on a mirror manifold of G_2 holonomy, then the mirror manifolds admit supersymmetric \mathbf{T}^4 (\mathbf{T}^3) fibrations; if Type IIA (or IIB) theory on a manifold of $Spin(7)$ holonomy has a mirror, then the mirror manifolds admit supersymmetric \mathbf{T}^4 fibrations.

Following our previous work [4], we have given examples of all three of the above types of mirror transformations which apply to fairly large classes of target spaces of exceptional holonomy. These transformations are further evidence for the above conclusions. Furthermore, in the last section we showed that all the target spaces in this class of examples have the above fibration structure; moreover, the relevant wrapped p -brane moduli space does so also.

Combining these results with those of [10, 13] we can see that there is a general picture in which manifolds which admit mirrors admit supersymmetric fibration by n -tori, where n varies from example to example.

We believe that mirror symmetry for manifolds of exceptional holonomy certainly deserves further study; both in string theory and mathematically. In the latter area, perhaps some more rigorously understood examples can be constructed along the lines of the construction in [25]. In the case of string theory, mirror symmetry for Calabi-Yau threefolds is defined perturbatively at the level of conformal field theory. In the case of Calabi-Yau target spaces the mirror symmetry is directly related to the representations of the superconformal algebra that underlies the string theory propagation on such spaces.

In [2] a generalised mirror conjecture was made. Applied to superstrings on manifolds of exceptional holonomy, this conjecture is related to the superconformal algebras associated with string propagation on target spaces of exceptional holonomy. As noted in [4] the mirror symmetries discussed here provide examples for which the generalised mirror conjecture holds true. It may be that a more precise definition of exceptional mirror symmetry could be given at the level of the superconformal field theories written down in [2], although it is not at all obvious to us if this is possible.

Finally we would like to mention that evidence for \mathbf{T}^3 fibered manifolds of G_2 holonomy has recently appeared in [26] in the context of constructing

the U -manifolds of [27], although the precise details of this construction have not been given.

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